S-Matrix Constraints on Effective Field Theories

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This is a brief review of S-matrix constraints on effective field theories. We will discuss such constraints from two possible avenues, 1. The assumption of a UV complete theory that satisfies the usual S-matrix axioms. 2. The low energy effective field theory as a spontaneous symmetry broken phase of the UV theory. In both cases, constraints on higher dimensional operators are found. We will also discuss its fate under quantum corrections.

1 Introduction

Effective field theory (EFT) has long been an effective tool for phenomenological purposes. In general we write the lagrangian as

\[ \mathcal{L} = \mathcal{L}_{\text{Marginal}} + \sum_{\Delta, i} a_{\Delta, i} \mathcal{L}_{\Delta, i} \]  

(1)

where \( \mathcal{L}_{\text{Marginal}} \) is the renormalizable part of the action with only marginal coupling, and \( \mathcal{L}_{\Delta, i} \) are the higher dimensional irrelevant operators with dimensions \( \Delta \) and labelled by \( i \). Naively one would expect that besides low energy global symmetries, there are no constraints on the coefficients of \( a_{\Delta, i} \). However, if we assume that the EFT has a UV completion that respects Lorentz invariance and the standard analyticity constraints that descends from unitarity, then there exists positivity bounds on some of the four-point operators [1]. Note that such constraints do not arise from any low energy pathology such as unbounded potentials. The implementation of such constraint can have powerful restriction once RG running is taken into account, since no where along the RG flow can the bounds be violated. This precisely led to the proof of the famous \( \alpha \)-theorem in four-dimensions [2].

The non-linear symmetries leave their fingerprints as soft theorems for S-matrix element. It is well known that amplitudes involving pions from SBS would vanish as the pion momenta becomes soft [4]. In [5], it was shown that the algebraic structure of the coset group reveals itself in the double soft limit. For spacetime symmetries, such as conformal symmetry, one can show that spontaneous symmetry breaking (SSB) leads to leading and subleading soft theorems [6, 7]. Through recursion relations, it was shown that such soft theorems is sufficient to fix a large part of the tree-level S-matrix up to a finite set of operators [8]. Note that knowing the tree-level S-matrix is equivalent to obtaining the lagrangian of the EFT.

Note that for CFTs that are isolated fix points, there are generically no lagrangian description. However, for those who have a vacuum manifold, the low energy theory does have a lagrangian description and is given by the Goldstone bosons and fermions of broken conformal, special super-symmetry and R-symmetry. Thus the information of the interacting theory is fully contained by the coefficients of these higher-dimensional operators. Thus the development of the complete soft-theorems, including leading and all subleading result, for these symmetries will fa-
cilitate in our carving out the space of consistent conformal theories, similar to the recent progress in conformal bootstrap [9].

2 Positivity Bounds on Higher-Dimensional Operators

Let’s consider the four-point S-matrix of identical scalars in a EFT. Since there are no three-point on-shell amplitude, the four point amplitude must be a polynomial of Mandlestam variables \( s, t \), where \( s \equiv (p_1 + p_2)^2 \) and \( t \equiv (p_1 + p_4)^2 \). This implies that in the forward limit, \( t \to 0 \), the four point amplitude is given as

\[
A_4(s, 0) = \sum_{i=0}^{\infty} f_i s^{2i}
\]

the fact that only even power of \( s \) remains in the forward limit is due to Bose symmetry of identical particles \( A_4(s, t) = A_4(u, t) \), which in the forward limit becomes \( A_4(s, 0) = A_4(-s, 0) \). The coefficient of higher dimensional operators are encoded in the coefficients \( f_i \). The analyticity of S-matrix and hence allows us to analytically continue the function \( A_4(s, 0) \) to complex \( s \). Then the coefficients \( f_i \) can be represented as

\[
f_i = \int_{C_1} ds \frac{A_4(s, 0)}{s^{2i+1}},
\]

where the integration contour encircles the pole at the origin. Since the only singularities or branch cuts must lie on the real axes, this implies that we can deform the contour \( C_1 \) into contour \( C_2 \), shown in Fig. 1. Contour \( C_2 \) probes the structure of the S-matrix in the UV, and hence with the usual assumption of analyticity of the S-matrix, only branch cuts or poles associated with the presence of UV degrees of freedom can appear. Note that this deformation requires that the branch cuts (or the singularities) do not reach the origin, which occurs when there are no fundamental three-point massless interactions. The behaviour of the amplitude at \( s \to \infty \) should behave as \( s^a \) with \( a < 1 \), in order for the UV completion to actually have a better behaviour at high energies compared to the infrared theory. This implies that the part of the contour at infinity does not contribute, and one is left with

\[
f_i = \int_{s_0}^{\infty} ds \frac{D \text{is}[A_4(s, 0)]}{s^{2i+1}} = \int_{s_0}^{\infty} ds \frac{\sigma}{s^{2i+1}},
\]

where \( \sigma \) is positive [1].

The above discussion applies to operators that are identical scalars. Similar bounds have been derived for operators with spins in the study of gravity and WW interaction, as well as for fermions [10]. As an example of its application, consider the case of RG flows in four-dimensions between two fixed points. In the IR, one has the dilaton effective action with the form,

\[
L = \delta a (\partial \phi)^2
\]

where \( \phi \) is the dilaton. The remaining terms are higher derivative corrections. \( a \) is the conformal anomaly, and \( \delta a \) is the difference of the conformal anomaly between the UV and IR CFT. Since \( E_4 \) is the only four derivative term in the action, this implies that positivity bounds can be directly applied to \( \delta a \), i.e. the difference between the UV and IR conformal anomaly must be strictly positive.

For non-identical scalars, one instead obtains positivity bounds on sums of higher dimensional operators. A systematic way of extracting the strongest bounds have been given in [11]. When applied to the corrections of Einstein-Hilbert gravity, one can again obtain dimension dependent positive bounds on sums of terms that are quadratic and quartic in curvatures, since the independent structures are sensitive to dimensions [12]. For the Gauss-Bonnet term there will be mixing with the ordinary Einstein-Hilbert contribution, which is divergent in the forward limit. Using an infrared regulator as well as assuming weak gravity regime, one can find a region in the parameter space where the Gauss-Bonnet term is dominant. In this region the Gauss-Bonnet term can introduce positive or negative sign for the forward limit contribution, depending on the choice of polarization vectors, and hence violates unitarity. This is in agreement with the analysis of [13], which showed that the presence of a Gauss-Bonnet term in the weakly coupled regime can cause acausal effects, which can only be cured by the presence of a set of infinite higher-spin massive states.
3 Soft Theorems and Spontaneous Symmetry Breaking

EFT describes systems where the degrees of freedom are light compared to the UV cutoff. Often this requires a mechanism where the these degrees of freedom can be naturally light. For vectors and spin-1/2 particles, this mechanism is gauge and chiral symmetry. For scalars, the most commonly utilised mechanism is spontaneous symmetry breaking. While the induced non-linear transformations on fields depends on the nature of the symmetry breaking, their fingerprint on S-matrix elements can be formulated in a universal fashion. This fingerprint is the behaviour of the S-matrix when ever the momenta of the Goldstone Boson becomes soft, the S-matrix factories into soft operators acting on lower-point S-matrix. Depending on the symmetry in question, the soft operator may include both leading and subleading terms, and they are universal in that they hold true irrespective of the multiplicity.

To see how soft theorems arrises, one simply starts with the Ward identity:

\[ q^\mu \langle J^\mu (q) \phi (p_1) \cdots \phi (p_n) \rangle = \sum_{i=1}^{n} \langle \cdots \delta \phi (p_i + q) \cdots \rangle, \]

where \( J \) is the current of the broken generator, Fourier transformed. Applying LSZ reduction on the \( n \)-legs on both side, the LHS contains the amplitude of a GB, with momenta \( q \), with \( n-\phi \) fields. Taking the limit \( q \to 0 \) isolates this term on the LHS by isolating the pole term as the current operator acts on the vacuum. On the RHS, taking \( q \to 0 \) allows non-vanishing result after the LSZ reduction. In particular if the broken symmetry transform the field \( \phi \) into a physical field, then the RHS is non-vanishing. If it is such that \( \delta \phi \) is not a physical field, then the RHS is zero. The later case is what occurs for general cosets \( G/H \) and when \( \phi \) is also a Goldstone boson. In this case, \( \phi \) is also given as the current associated with the broken generators \( T \), the broken group generators, and since

\[ [T, T] \sim H \]

where \( H \) is that of the invariant subgroup, which does not create a physical state, the RHS is zero. This is the famous Adler’s zero [4]. In the following we will consider cases when the RHS is not zero.

Note that double-soft theorems can also be obtained by considering

\[ q_1^\mu q_2^\nu \langle J^\mu_1 (q_1) J^\nu_2 (q_2) \phi (p_3) \cdots \phi (p_n) \rangle \]

\[ = \sum_{i=3}^{n} \langle \cdots \delta ^\mu \delta ^\nu \phi (p_i + q_1 + q_2) \cdots \rangle \]

\[ + \sum_{i,j=3}^{n} \langle \cdots \delta ^\mu \phi (p_i + q_1) \cdots \delta ^\nu \phi (p_j + q_2) \cdots \rangle \]

In general the double soft limits are non-vanishing. For example, for Non-linear Sigma
models, we have
\[ M_n(\phi^{a_1}, \phi^{a_2}, \cdots)\big|_{p_1,2 \to p_{1,2}} = g^2 \sum_{i=3}^{n} B_i f^{a_1 a_2 a_i} M_{n-2}(\cdots, \phi^{a_i}, \cdots) + O(g^2) \]
where \( B_i \equiv \frac{p_i(p_1 - p_2)}{2p_i(p_1 + p_2)} \), and \( f^{a_1 a_2 a_i} \) is the structure constant in
\[ [T^{a_1}, T^{a_2}] = f^{a_1 a_2 a_i} H^{a_i}. \] (9)
Similar results can be obtained when the coset is a supergroup [14].

3.1 Spacetime Symmetry Breaking

Cases where the broken symmetries are spacetime symmetries are prime examples where the single soft theorems can yield non-vanishing results. This is due to the fact that the Goldstone Boson can be associated with multiple broken generators. Consider spontaneous breaking of conformal symmetry, where both the dilatation and conformal boost generators are broken. There is only one Goldstone Boson, the dilaton, since the mode associated with broken conformal boosts can be derived from that of broken dilatation operator, through the algebra
\[ [K, P] \sim D. \] (10)
This also implies, that the RHS of Eq.(6) does not vanish after LSZ reduction.

The broken dilatation symmetry constrains the leading term whilst the conformal boost generators constrain the sub-leading term in the \( q \) expansion of Eq.(6). Thus amplitudes with single soft dilaton (\( \phi \)) satisfy the following universal soft theorem [6, 7]:
\[ v A_{n}|_{p_n \to 0} = \left( S^{(0)}_n + S^{(1)}_n \right) A_{n-1} + O(p_n^2), \] (11)
where the superscript indicates the degree in \( q \) and \( v \) is the vacuum expectation value of the dilaton field. The explicit form of \( S^{(0)}_n, S^{(1)}_n \) are given by
\[ S^{(0)}_n = -\sum_{i=1}^{n-1} \left( p_i \frac{\partial}{\partial p_i} + \frac{D-2}{2} \right) + D, \]
\[ S^{(1)}_n = -\rho^2 \sum_{i=1}^{n-1} \left[ p_i^r \frac{\partial^2}{\partial p_i^r \partial p_i^r} - \frac{p_{ij}^r}{2} \frac{\partial^2}{\partial p_i^r \partial p_j^r} \right] + \frac{D-2}{2} \frac{\partial}{\partial p_i^r}. \]
where \( D \) is the space-time dimension. Note that for theories which don’t have odd point amplitudes, this would predict that the soft limit vanishes as \( g^2 \) in the soft-moments expansion. Indeed this accounts for the recently discussed enhanced soft limits of DBI action [Cliff].

For superconformal theories, further global symmetries are broken, including (fermionic) conformal supersymmetry and R-symmetry. For instance, in \( N = 4 \) SYM, the scalars form a 6 of R-symmetry SO(6). If any one of the scalars taking a vev (say \( \phi^6 \)) breaks R-symmetry down to SO(5), with 5 GB’s associated with the broken rotation generators \( R^{6I} \) with \( I = 1, \cdots, 5 \). Under this broken generator, the GB’s \( \phi^I \) is rotated into \( \phi^6 \equiv \varphi \), while \( \varphi \) is rotated into \( \phi^I \) with a relative minus sign due to the antisymmetry of \( R^{6I} \). Thus the soft limit of R-symmetry GB’s are given as:
\[ v A_n(\phi_1, \cdots, \phi^6_n)|_{p_n \to 0} = \sum_i A_{n-1}(\cdots, \delta I \phi_i, \cdots) + O(p_n^2), \] (12)
where \( \phi_i \) represents either a dilaton \( \varphi \) or \( \phi^I \), with \( \delta I \varphi = \phi^I \) and \( \delta I \phi^I = -\delta I^I \varphi \). Thus the six massless scalars for the \( N = 4 \) theory are all Goldstone Bosons, while the fermions correspond to “goldstinos” of the 16 broken conformal busy generators.

3.2 Recursive Determination of the S-Matrix

Note that the knowledge of the tree-level S-matrix is equivalent to obtaining the action of the effective theory. In fact, it is advantageous in the sense that it is free of field redefinition ambiguities. Thus the question becomes if the knowledge of soft theorems can be utilised to construct the full S-matrix.

Recall that the well established Britto, Cachazo, Feng and Witten (BCFW) recursion relation uses the fact that the amplitude have another well known universal behaviour, on factorisation poles the residue is given by the product of lower point amplitudes. One start with [15, 16]:
\[ A_n(0) = \frac{1}{2\pi i} \oint C_0 dz \frac{A_n(z)}{z}, \] (13)
where the contour \( C_0 \) encircles the origin, and \( A_n(z) \) is the \( n \)-point amplitude with deformed
momenta. As $A_n(z)$ is meromorphic, via the residue theorem we can recast the amplitude, which is simply $A_n(0)$ at the origin, as a sum over residues of the factorisation poles plus the one at infinity.

Note that all that is required is universal behaviour, and to utilise soft theorems, one simply needs to introduce a new rational function whose poles forces the kinematic of the deformed amplitude to be in the soft limit. For example consider

$$A_n(0) = \frac{1}{2\pi i} \oint_{C_0} dz \frac{A_n(z)}{zF(z)}.$$  \hspace{2cm} (14)

where $F(z)$ is a polynomial in $z$ with $F(0) = 1$, and its zeroes correspond to the special kinematic configurations. At large $z$, $F(z) \sim z^d$ with some positive $d$, and thus the function $F(z)$ introduces extra power of suppression at large $z$, allowing for vanishing boundary contributions for theories with higher dimensional operators. The amplitude $A(0)$ is then determined by the residues of the factorisation pole as well as the contributions from the poles in $1/F(z)$ which are given by the universal behaviour of the amplitudes.

A construction that achieves this was recently introduced by [17] for cases where the soft theorems simply vanishes. The deformation on the amplitude is given as $p_i \rightarrow (1-za_i)p_i$, while $F(z)$ is

$$F_n(z) = \prod_{i=1}^{n} (1-za_i)^{d_i}.$$  \hspace{2cm} (15)

As one can see, the new poles at $z \rightarrow 1/a_i$ corresponds to the soft limit for the shifted momenta. To ensure momentum conservation, we have $\sum_i a_i p_i = 0$. Note that this can be straightforwardly extended to cases where the soft limit does not vanish, as in this case one simply adds the residues for the poles in $1/F_n(z)$. The degree $d_i$ reflects to which order the soft theorem is valid, for example for theories with only leading universal soft theorems $d_i = 1$, while subheading implies $d_i = 2$.

Note that Eq.(14), yields a recursive construction of the $n$-point amplitude in terms of lower point amplitudes only if there are no contributions at $z = \infty$. The behaviour at $z = \infty$ can be estimated by the mass dimension of the amplitude: for an amplitude with mass dimension $s^n$, one can have at most $z^{2p}$. Thus a recursion formula would be valid if $\sum_i d_i = k \geq 2p + 1$. For dilaton effective action, this implies that for $s^p$ amplitudes, all amplitudes beyond $p$-points are completely determined by lower point amplitudes. One should also take into account that the soft BCFW recursion relation is only applicable in $D$-dimensions for at least $D+2$ external legs, in order for a valid shift to be constructed. For instance, in $D = 4$, at order $s^4$, knowing the four-point amplitude is not enough to completely fix all higher-point amplitudes. Instead the five-point amplitude is required to fully determine all amplitudes at this order. This general discussion is summarized in Table 1. For general superconformal theories, one can consider mixed amplitudes with $n_1$ dilatons and $n_2$ R-symmetry
Goldstone bosons. Since the R-symmetry soft theorem is only leading, the requisite bound for valid recursion is $n_1 + \frac{1}{2}n_2 > k$ for order-$s^k$ amplitudes.

With supersymmetry, further constraint can be imposed. As shown in [8], with maximal supersymmetry, the effective action is fixed up to eight derivatives in terms of just one unknown four-point coefficient and one more coefficient for ten-derivative terms. The validity of soft theorems both in the UV where the massive degrees of freedom is present or the IR where they are integrated away have been verified for the Coloumb branch effective action of $\mathcal{N} = 4$ SYM.

4 Conclusions

In this review, we’ve discussed various constraint on the coefficients of higher dimensional operators both from unitarity constraints derived from the UV complete theory, as well as the non-linear symmetries that reflects the IR theory emerging from spontaneous symmetry breaking in the UV. There are many interesting avenues to pursuit. So far, all UV unitarity constraint has been derived by considering the forward limit of 2 to 2 scattering, it would be extremely interesting if the analysis can be pushed to 3 to 3 forward scattering. This may let us gain insight for the interplay between scale vs conformal symmetry. As discussed the breaking of dilatation and conformal boost is what accounts for leading and subheading soft theorems. For a theory with only scale but not conformal invariance, this would mean that in the IR one would no longer have subheading soft theorems. Interestingly the leading soft theorems fixes the five point S-matrix in such a way that the subheading soft is automatically satisfied [8]. The first case where subheading soft is violated is at six-points. Thus knowing the six-point unitarity constraints may lead us to understand why such violation at subheading order is a reflection of non-unitarity in the UV.

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References


